From Theories to Categories

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In Sets...

We prove using elements.

THEOREM. If a monoid has inverses, these are unique.

PROOF. Consider x, y inverses to a; then

$$x = x \cdot e = x \cdot (a \cdot y) = (x \cdot a) \cdot y = e \cdot y = y. \qquad \square$$

Elsewhere

However, this is true in categories other than Sets.

A monoid in a cartesian multicategory is a pair $m : A \times A \to A$ and $u : 1 \to A$ with

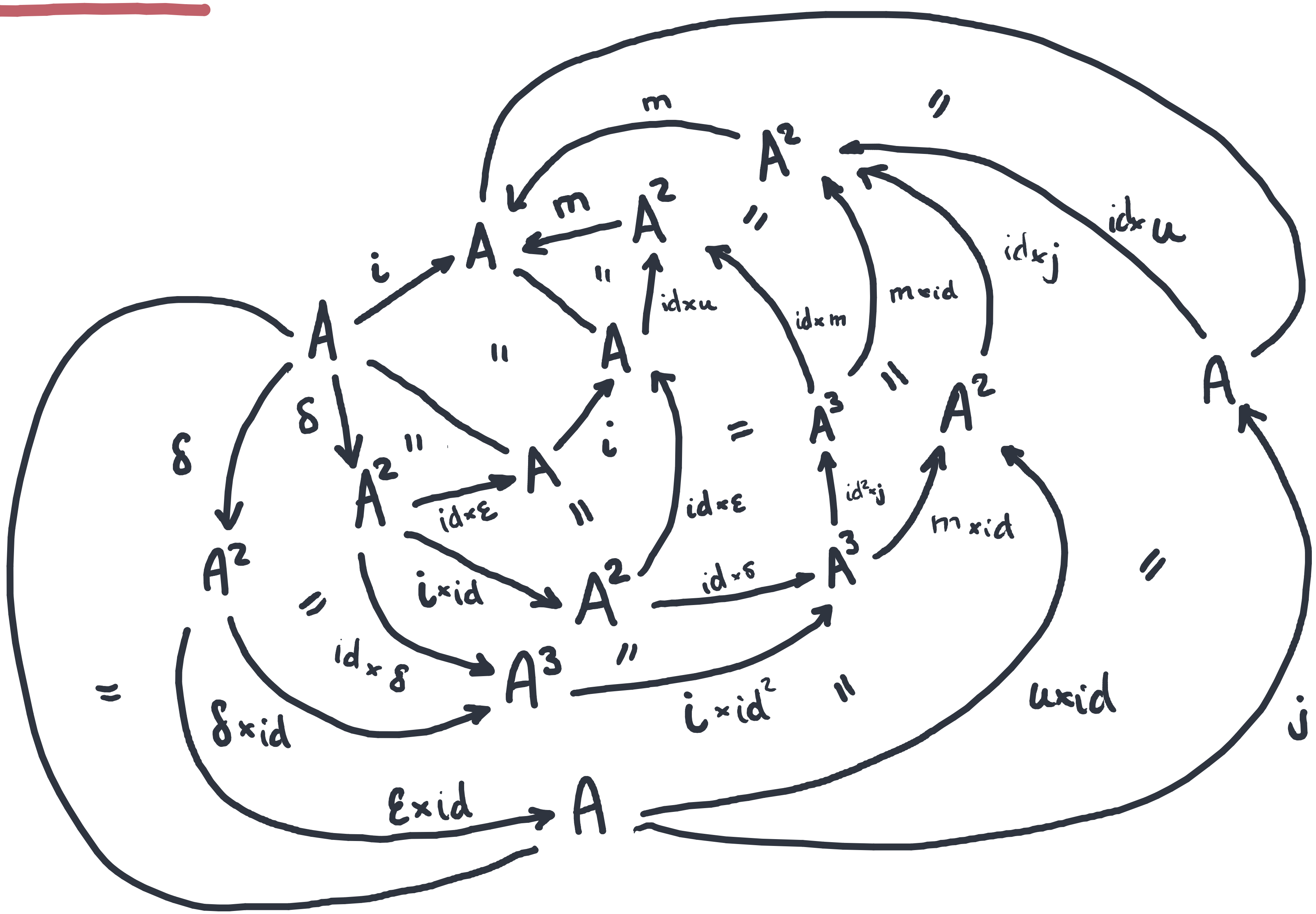
$$
\begin{tikzcd}
A^3 \arrow[r, "id \times m"] \arrow[d, "m \times id"'] \arrow[dr, phantom, "="] & A^2 \arrow[d, "m"] \\
A^2 \arrow[r, "m"] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
 & A \arrow[dl, "1 \times u"'] \arrow[d, "id"] \arrow[dr, "u \times 1"] & \\
A \times A \arrow[r, "m"] & A & A \times A \arrow[l, "m"']
\end{tikzcd}
$$

An **antipode** is a morphism $A \xrightarrow{i} A$ such that

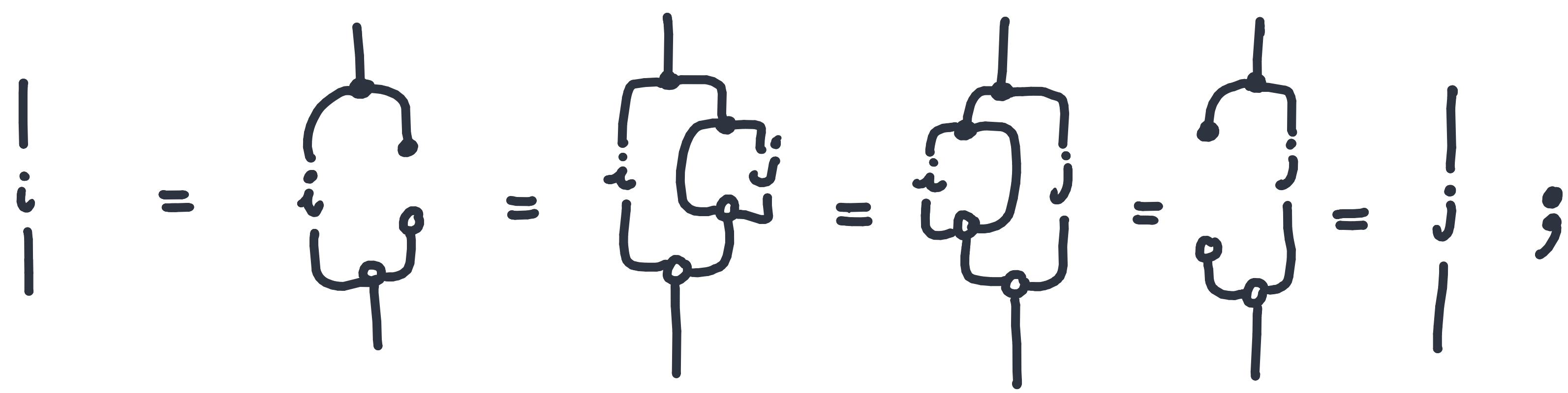
$$
\begin{tikzcd}
 & A \times A \arrow[r, "i \times id"] & A \times A \arrow[dr, "m"] & \\
A \arrow[ur, "\delta"] \arrow[r, "!"] \arrow[dr, "\delta"'] & 1 \arrow[r, "u"] & & A \\
 & A \times A \arrow[r, "id \times i"'] & A \times A \arrow[ur, "m"'] &
\end{tikzcd}
.
$$

THM. Antipodes are unique.

PROOFS ARE LEFT TO THE READER



AT LEAST



As with other algebraic structures there are again two aspects, the structures themselves and their presentations which are closely related, yet quite distinct; for example, more than one presentation may be needed for efficient calculations determining features of the same algebraic structure.

A Better Way

Surely, we were just translating the proof from Sets. Cannot we just repeat it?

$$x : X \vdash \; i(x) = i(x) \cdot e = i(x) \cdot (x \cdot j(x)) = (i(x) \cdot x) \cdot j(x)$$
$$= e \cdot j(x) = j(x).$$

This is true for generalized elements outside Set.

In his autobiography[5]*, Mac Lane writes that "One day, Sammy told me he had a young student who claimed that he could do set theory without elements. It was hard to understand the idea, and he wondered if I could talk with the student. (...) I listened hard, for over an hour. At the end, I said sadly, 'Bill, this just won't work. You can't do sets without elements, sorry,' and reported this result to Eilenberg. Lawvere's graduate fellowship at Columbia was not renewed, and he and his wife left for California." ...*

... I never proposed "Sets without elements" but the slogan caused many misunderstandings during the next 40 years because, for some reason, Saunders liked to repeat it. Of course, what my program discarded was instead the idea of elementhood as a primitive, the mathematically relevant ideas of both membership and inclusion being special cases of unique divisibility with respect to categorical composition. I argue that set theory should not be based on membership, as in Zermelo-Frankel set theory, but rather on isomorphism-invariant structure.

An Interview with W. Lawvere. Jorge Picado.

THEORY OF CARTESIAN MULTICATS.

Over any multigraph of generators, we construct a theory.

$$\frac{}{\Gamma \ni x : X \vdash x : X} \qquad \frac{\Gamma \vdash t_0 : A_0 \quad \cdots \quad \Gamma \vdash t_n : A_n}{\Gamma \vdash f(t_0, \dots, t_n) : B}(f)$$

THM. Derivations of $A_0, \dots, A_n \vdash B$ are the morphisms of the free cartesian multicategory over the generators.

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THM. Derivations of $A_0, \dots, A_n \vdash B$ are the morphisms of the free cartesian multicategory over the generators.

$$\begin{tikzcd} \mathrm{MultiGraph} \arrow[r, bend left, "\text{terms}"] \arrow[r, phantom, "\perp"] & \mathrm{CartMult} \arrow[l, bend left, "\text{forget}"] \end{tikzcd}$$

We quotient by equations to get the "walking" theory.

WHY CARTESIAN MULTICATEGORIES

A cartesian multicategory is a category with specified finite products, whose set of objects under the product operation is a free monoid on specified generators.

Cartesian multicategories are colored Lawvere theories.

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$$x : X \vdash \quad i(x) = i(x)\cdot e = i(x)\cdot\big(x\cdot j(x)\big) = \big(i(x)\cdot x\big)\cdot j(x)$$
$$= e\cdot j(x) = j(x).$$

This holds in the free cartesian-with-a-monoid, hence in any monoid in a cartesian multicat.

In all those areas where category theory is actively used the categorical concept of adjoint functor has come to play a key role. Such a universal instrument for guiding the learning, development, and use of advanced mathematics does not fail to have its indications also in areas of school and college mathematics, in the most basic relationships of space and quantity and the calculations based on those relationships. By saying "take categories seriously", I meant that one should seek, cultivate, and teach helpful examples of an elementary nature.

An Interview with W. Lawvere. Jorge Picado.

STRINGS?

Analogously,

$$\mathrm{MultiGraph} \underset{}{\overset{\mathrm{Terms}}{\rightleftarrows}} \mathrm{CartMult} \; ; \qquad \mathrm{PolyGraph} \underset{}{\overset{\mathrm{String}}{\rightleftarrows}} \mathrm{MonCat} .$$

String diagrams allow to compute with theories on the doctrine of monoidal categories.

E.G. The augmented simplex (infinite presentation as a category) is the walking monoid.

And 2-Dimensionally

$$
\begin{tikzcd}
\mathrm{PolyGraph} \arrow[rr, bend left, "\mathrm{String}"] & \mathrm{MonCat} \arrow[l, "\mathrm{Forget}"] & \mathrm{StrictMonCat} \arrow[l, "\mathrm{Forget}"]
\end{tikzcd}
$$

There is a 2-adjunction between Polygraphs and Monoidal Cats.
It arises from the strict one and coherence, $\mathrm{Str}(M) \simeq M$.

And 2-Dimensionally

$$
\begin{tikzcd}
\mathrm{PolyGraph} \arrow[r, bend left] \arrow[r, phantom, "\perp"] \arrow[rr, bend left=50, "\mathrm{String}"] \arrow[rr, phantom, bend left=25, "\perp"] & \mathrm{MonCat} \arrow[l, bend left, "\mathrm{Forget}"] \arrow[r, bend left, "\mathrm{Strictify}"] \arrow[r, phantom, "\perp"] & \mathrm{StrictMonCat} \arrow[l, bend left, "\mathrm{Forget}"]
\end{tikzcd}
$$

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