Notes on "A Graphical View"

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Tallcat reading seminar.

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PART 1 : PREMONOIDAL

Noidal Categories

DEFINITION. A noidal category is a category $(\mathcal{O}, \hom)$ endowed with a map on objects $(\otimes)\colon \mathrm{List}(\mathcal{O}) \to \mathcal{O}$, which must be independently functorial on each component.

$$(\otimes)\colon \mathcal{O}^n \longrightarrow \mathcal{O} \quad \text{is a sesquifunctor, and} \quad \otimes(A) = A.$$

The centre of a noidal category, $Z(\mathcal{O}, \hom)$, is formed by the morphisms that interchange; that is, $f\colon A_i \to A_i'$ is central if, for each $g\colon A_j \to A_j'$,

$$\begin{tikzcd}
 & \otimes A_k^{i} \arrow[dr, "g"] \arrow[d, phantom, "\shortparallel"] & \\
\otimes A_k \arrow[ur, "f"] \arrow[dr, "g"'] & {} & \otimes A_k^{i,j} \\
 & \otimes A_k^{j} \arrow[ur, "f"'] &
\end{tikzcd}.$$

Premonoidal Categories

DEFINITION. A premonoidal category is a noidal category such that for each $\ell \in \mathrm{LIST}(\mathrm{LIST}(\mathbb{C}))$, there exists a central isomorphism

$$\alpha : \otimes(\mathrm{concat}(\ell)) \to \otimes(\mathrm{map} \otimes \ell)$$

that is separately natural on each component and such that any formal equation with α, id holds true.

EXAMPLE. For instance, if $\ell = [[A,B],[C],[D,E,F]]$,

$$\alpha_\ell : A \otimes B \otimes C \otimes D \otimes E \otimes F \to (A \otimes B) \otimes C \otimes (D \otimes E \otimes F).$$

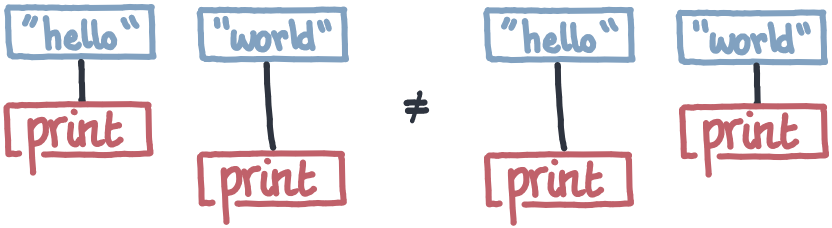
FREYD CATEGORIES

DEFINITION. A freyd category is an identity-on-objects functor from a symmetric monoidal category V ("the values") to a symmetric premonoidal category C ("the computations"), strictly preserving the premonoidal structure.

EXAMPLES. The kleisli category of a strong monad is a Freyd category. In fact, any Freyd category arises from a colimit preserving monad on the presheaf cocompletion.

COLORING PREMONOIDALITY

In a freyd category $V \to C$, we can use two different colours to declare if we are depicting a value or a computation.



Here, print does not need to satisfy the interchange law. Jeffrey proposes to represent this as another control wire that needs to be passed around to all the computations.

Coloring Premonoidality

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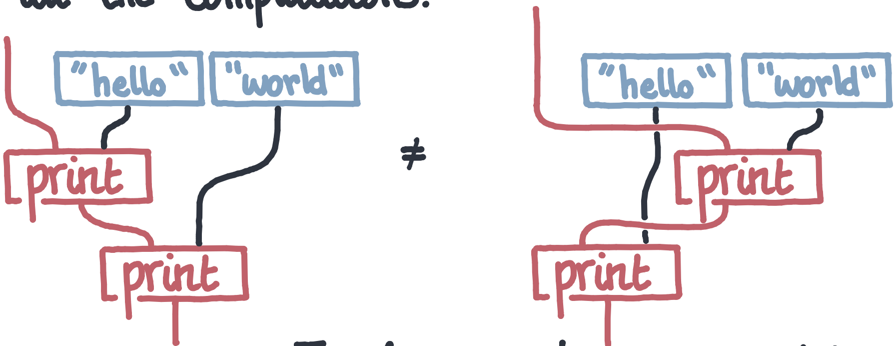


CONJECTURE. The free symmetric premonoidal category on some generators is the free symmetric monoidal category on the same generators enhanced with a control wire that is input and output of every morphism.

"Runtime as a resource."

Part 2 : Diagrams are programs

Arrow-Do Notation

Morphisms $A_1 \otimes \cdots \otimes A_n \to B_1 \otimes \cdots \otimes B_m$ of the free sym. premonoidal cat. over a multigraph $\mathcal{G}$ are Arrow blocks.

proc $(x_0, \dots, x_n) \to$ do
$\quad x^1_1, \dots, x^1_{n_1} \leftarrow g_1 -\!\!\prec y^1_0, \dots, y^1_{m_1}$
$\quad \vdots$
$\quad x^k_1, \dots, x^k_{n_k} \leftarrow g_k -\!\!\prec y^k_0, \dots, y^k_{m_k}$
return $(y_0, \dots, y_m)$

Where each y must be an x that appeared before. Each x must appear exactly once.

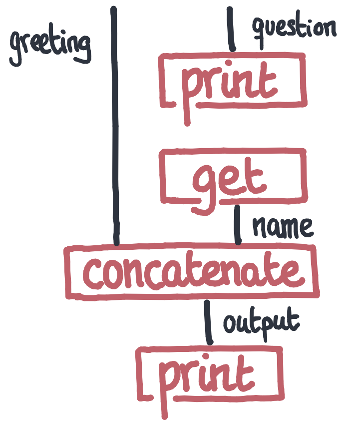
$g_1, \dots, g_k \in \mathcal{G}$ are generators.

And the types must match.

Arrow-Do Notation

EXAMPLE. HelloProgram

```
proc (question, greeting) -> do
    () <- print -< question
    name <- get -< ()
    output <- concatenate -< greeting, name
    () <- print -< output
return ()
```



RUNNING HelloProgram ("What's your name?", "Hi, ")

```
What's your name?
 > Mario
Hi, Mario.
```

Type Theory of Symmetric Monoidal Cat.

This was a language for computations. What about values?

Morphisms $A_1 \otimes \cdots \otimes A_n \to B_1 \otimes \cdots \otimes B_m$ of the free sym. monoidal cat. over a multigraph $\mathcal{G}$ are terms of the following type theory.

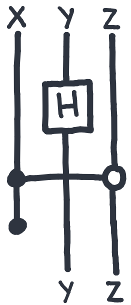
$$\frac{}{x : A \vdash x : A}\ \text{VAR} \qquad \frac{f \in \mathcal{G}(A_1, \ldots, A_n ; B) \quad \Gamma_1 \vdash x_1 : A_1 \quad \ldots \quad \Gamma_n \vdash x_n : A_n}{\Gamma_1, \ldots, \Gamma_n \vdash f(x_1, \ldots, x_n) : B}\ f$$

$$\frac{\Gamma_1 \vdash x_1 : A_1 \quad \ldots \quad \Gamma_n \vdash x_n : A_n}{\mathrm{Shuf}(\Gamma_1, \ldots, \Gamma_n) \vdash [x_1, \ldots, x_n] : A_1 \otimes \cdots \otimes A_n}\ \text{TUPLE}$$

$$\frac{\Delta \vdash m : A_1 \otimes \cdots \otimes A_n \quad \Gamma, x_1 : A_1, \ldots, x_n : A_n \vdash z : C}{\mathrm{Shuf}(\Gamma, \Delta) \vdash [x_1, \ldots, x_n] \leftarrow m ; z : C}\ \text{SPLIT}$$

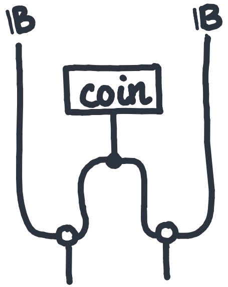
TYPE THEORY OF SYMMETRIC MONOIDAL CAT.

EXAMPLE. Quantum circuit.



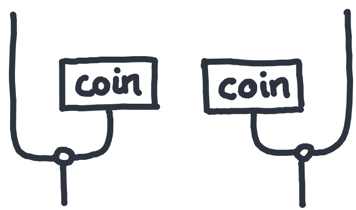
$[x_\circ, z_\circ] \leftarrow \mathrm{cnot}(x, z)$
$[\,] \leftarrow \mathrm{discard}(x_\circ)$
$[H(y), z_\circ]$

EXAMPLE. Sampling once or twice.



$[c_1, c_2] \leftarrow \mathrm{copy}(\mathrm{coin})$
$[b_1 \oplus c_1,\ b_2 \oplus c_2]$

$[b_1 \oplus \mathrm{coin},\ b_2 \oplus \mathrm{coin}]$



Arrow-Do Notation

Morphisms $A_1 \otimes \cdots \otimes A_n \to B_1 \otimes \cdots \otimes B_m$ of the free sym. Freyd cat. over a multigraph $\mathcal{G}$ are Arrow blocks with monoidal terms.

$$\begin{array}{l}
\mathbf{proc}\ (x_0, \dots, x_n) \rightarrow \mathbf{do} \\
\quad x^1_1, \dots, x^1_{n_1} \leftarrow g_1 \leftarrow E^1_0, \dots, E^1_{m_1} \\
\quad y^1_1, \dots, y^1_{e_1} \leftarrow E_1 \\
\qquad \vdots \\
\quad x^k_1, \dots, x^k_{n_k} \leftarrow g_k \leftarrow E^k_0, \dots, E^k_{m_k} \\
\quad y^k_1, \dots, y^k_{e_1} \leftarrow E_k \\
\mathbf{return}\ (E^0, \dots, E^m)
\end{array}$$

Where each E must use x's, y's that appeared before. Each x, y must appear exactly once.

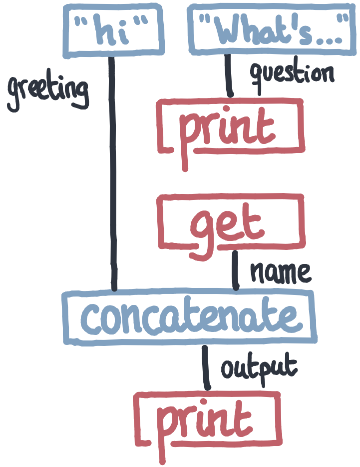
$g_1, \dots, g_k \in \mathcal{G}$ are generators.

And the types must match.

Arrow-Do Notation

EXAMPLE. HelloProgram

```
proc () → do
  question ← "What's your name?"
  () ← print ─< question
  name ← get ─< ()
  () ← print ─< concatenate("hi, ", name)
return ()
```

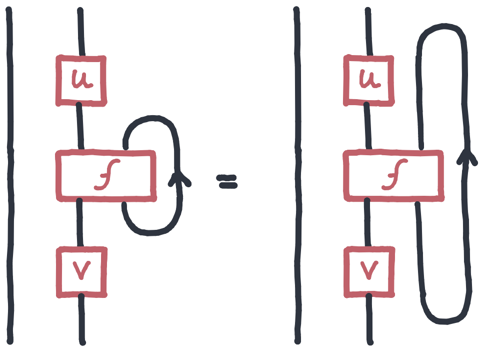


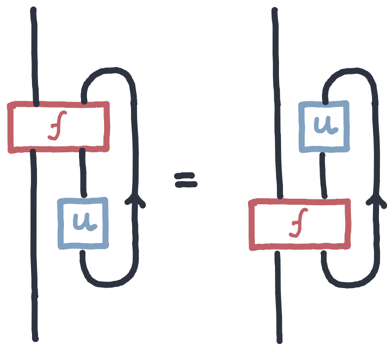
Feedback Premonoidal Category

Definition. A feedback premonoidal category is a freyd category $\mathcal{V} \to \mathcal{C}$ endowed with an operator

$$\mathsf{fbk}^{S} : \mathrm{hom}(S \otimes A, S \otimes B) \longrightarrow \mathrm{hom}(A, B)$$

that satisfies the following axioms.

1. 

2. 

3. $\mathsf{fbk}^{S_1}(\cdots \mathsf{fbk}^{S_n}(f) \cdots) = \mathsf{fbk}^{S_1 \otimes \cdots \otimes S_n}(f)$.

Arrow-Loop Notation

Morphisms $A_1 \otimes \cdots \otimes A_n \to B_1 \otimes \cdots \otimes B_m$ of the free feedback premon. cat. over a multigraph $\mathcal{G}$ are Arrow Loop blocks with monoidal terms.

$$
\begin{array}{l}
\mathbf{proc}\ (x_0, \dots, x_n) \to \mathbf{do} \\
\quad x^1_1, \dots, x^1_{n_1} \leftarrow g_1 \prec E^1_0, \dots, E^1_{m_1} \\
\quad y^1_1, \dots, y^1_{e_1} \leftarrow E_1 \\
\qquad \vdots \\
\quad x^k_1, \dots, x^k_{n_k} \leftarrow g_k \prec E^k_0, \dots, E^k_{m_k} \\
\quad y^k_1, \dots, y^k_{e_1} \leftarrow E_k \\
\mathbf{return}\ (E^0, \dots, E^m)
\end{array}
$$

Variables can appear even before defining them. Each x must appear exactly once.

$g_1, \dots, g_k \in \mathcal{G}$ are generators.

And the types must match.

Arrow-Loop Notation

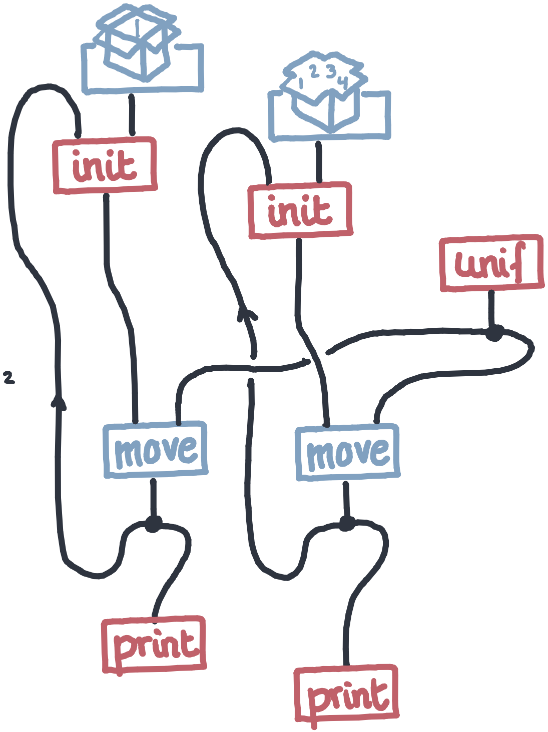
EXAMPLE. Ehrenfest.

proc () → do
  leftBox$_0$ ← init ≺ [], leftBox$_2$
  rightBox$_0$ ← init ≺ [1,2,3,4], rightBox$_2$

  ball$_{0,1}$ ← unif ≺ ()

  leftBox$_{1,2}$ ← move (ball$_0$, leftBox$_0$)
  rightBox$_{1,2}$ ← move (ball$_1$, rightBox$_0$)

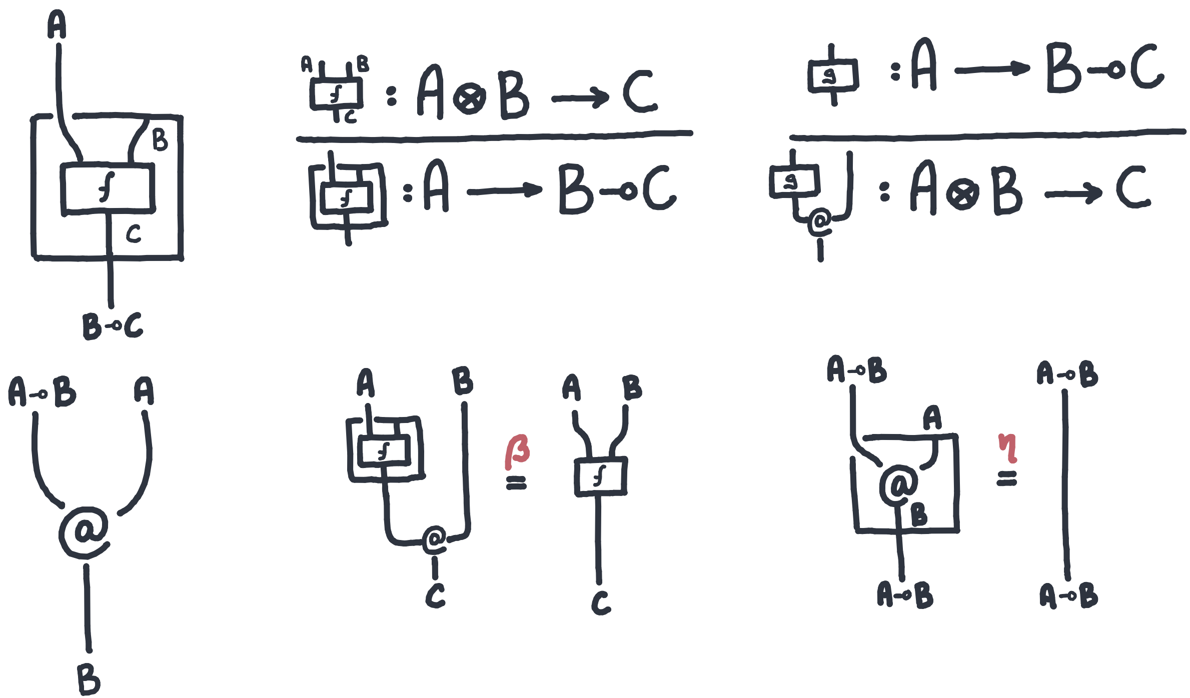
  () ← print ≺ leftBox$_1$
  () ← print ≺ rightBox$_1$
return ()



github.com/mroman42/arrow-streams

Part 3 : Closed Categories

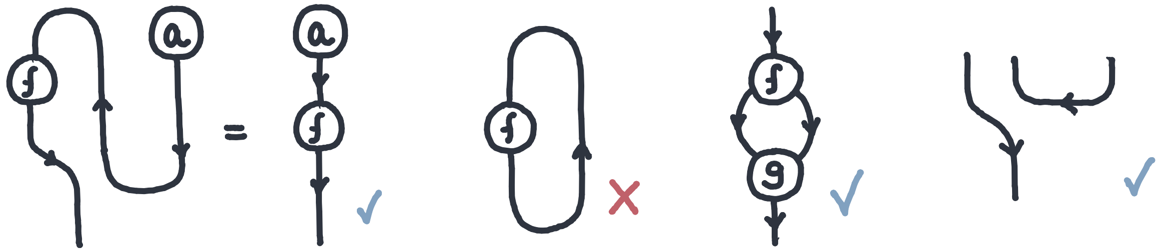
Closed Categories



Closed Categories, ∗-Autonomous Way

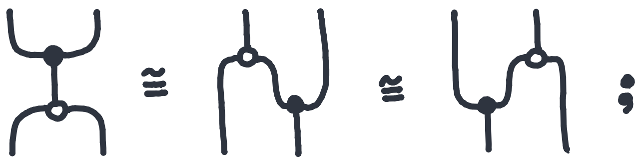
Symmetric monoidal closed categories fully-faithfully embed into ∗-autonomous categories, so we can use their graphical calculus.

- Every object has a formal dual $A\downarrow$ $A^*\uparrow$.
- Some diagrams are valid; some are not.
- $(A \multimap B) = A^* \oplus B$.

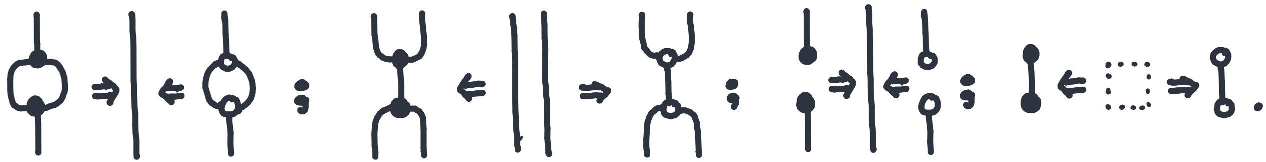


Closed Categories, ∗-Autonomous Way

A ∗-autonomous category is exactly a Frobenius monoid of multivariable adjunctions.



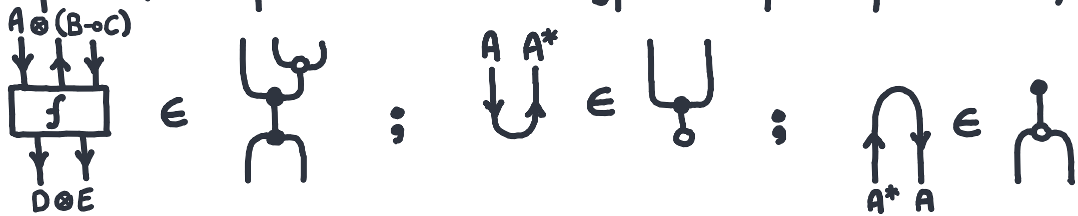
Frobenius monoid of representables.



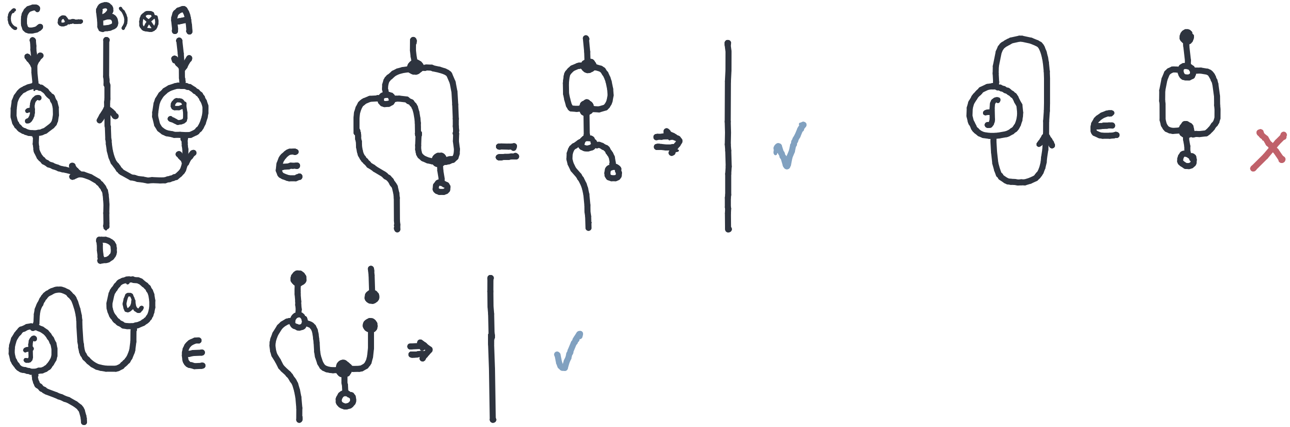
A diagram shape is valid if it can be reduced to the identity profunctor.

Closed Categories, *-Autonomous Way

Shapes of morphisms are their type. Cups/caps have fixed shapes.



Examples.



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