Commutative Tricocycloids

Mario Román
Tallinn University of Technology

Abstract

We introduce commutative tricocycloids: a model of parallel guarded choice inducing commutative monads based on the theory of tricocycloids developed by Street and Garner. Not every Set-based tricocycloid is commutative: we provide explicit counterexamples.

2012 ACM Subject Classification Theory of computation → Categorical semantics

Keywords and phrases Category theory

Digital Object Identifier 10.4230/LIPIcs.CVIT.2016.23

Funding *Mario Román*: supported by the Estonian Research Council grant PRG 3215, and by the Advanced Research + Invention Agency (ARIA) Safeguarded AI Programme.

Acknowledgements The authors thank Jack Liell-Cock, Mayuko Kori, and Kazuki Watanabe for discussion.

1 Tricocycloids

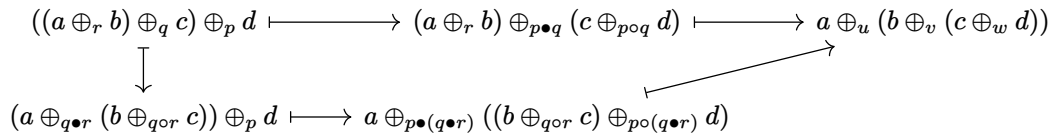
1.1 Motivation

Guarded choice algebras are based around a choice operation, $a \oplus_p b$, that chooses between two expressions, a and b, using a guard, p. The two expressions may be computations and contain guarded choices themselves; the guard may be probabilistic, possibilistic, or effectful.

Tricocycloids are a algebraic structure arising from quantum foundations and recently applied to guard algebras [3, 1]. The tricocycloid axioms derive from the associativity requirement of an algebra of guards. Assume the following rewriting axiom, which produces a terminating rewriting system:

$$(a \oplus_q b) \oplus_p c \mapsto a \oplus_{p \bullet q} (b \oplus_{p \circ q} c).$$

Then, to make the guards invariant to associativity rewriting, they must satisfy at least the pentagon equation: it is the equation coming from the single critical pair of this rewriting system.



Coherence is thus equivalent to the following three axioms.

1. $u = (p \bullet q) \bullet r = p \bullet (q \bullet r)$;
2. $v = (p \bullet q) \circ r = (p \circ (q \bullet r)) \bullet (y \circ z)$;
3. $w = p \circ q = (p \circ (q \bullet r)) \circ (y \circ z)$.

▶ **Definition 1** (Street [3]). *A* tricocycloid, $(H, \bullet, \circ)$, *consists of a set* H *endowed with two binary functions,* $(\bullet)\colon H \times H \to H$ *and* $(\circ)\colon H \times H \to H$, *that are jointly invertible — meaning that the function* $\varphi(p, q) = (p \bullet q, p \circ q)$ *is a bijection — satisfying the following axioms.*

1. $p \bullet (q \bullet r) = (p \bullet q) \bullet r$;
2. $(p \bullet q) \circ r = (p \circ (q \bullet r)) \bullet (q \circ r)$;

© Mario Román;
licensed under Creative Commons License CC-BY 4.0
Version for the 20 November 2026.
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

3. $(p \circ (q \bullet r)) \circ (q \circ r) = p \circ q$.

▶ Remark 2. We are saying that every tricocycloid induces a monoidal tensor

$$X \star_H Y = X + X \times H \times Y + Y.$$

We write the elements of this monoidal tensor as $x \in X$, as $y \in Y$, or as $x \oplus_p y \in X \times H \times Y$, depending on the case. We may define associativity and unitality structural maps in the only non-trivial case by $\alpha((x \oplus_q y) \oplus_p z) = (x \oplus_{p \bullet q} (y \oplus_{p \circ q} z))$. The pentagon and triangle equations hold by definition: we have shown this forms a monoidal category; conversely, if we have a monoidal category of this form, the pentagon equation forces H to be a tricocycloid.

▶ **Proposition 3** (Garner, Proposition 23). *A triple $(H, \bullet, \circ)$ forms a tricocycloid if and only if $(\star_H, 0)$ forms a monoidal category.*

1.2 Rewriting with tricocycloids

An (unlabelled) *binary tree* over consists either of a leaf, $\mathsf{leaf} \in T$; or a node, $\mathsf{node}(t_1, t_2) \in T$ for $t_1, t_2 \in T$; in other words, the set of unlabelled binary trees is the initial fixpoint of the functor $\mu A.1 + A \times A$.

▶ **Definition 4.** *Left $t \in T$ be an unlabelled binary tree with n nodes and $n+1$ leaves. We define the substitution of nodes and leaves on the tree as follows: for each list of elements $[x_0, ..., x_n]$ and each list of coefficients, $[p_1, ..., p_n]$, we define, an element of the free algebra over the tricocycloid, $\bigoplus_t^{\{p_1,...,p_n\}}\{x_0, ..., x_n\}$, by induction as follows.*

$$\bigoplus_{\mathsf{leaf}}^{\{\}}\{x\} = x; \qquad \bigoplus_{\mathsf{node}(t_1,t_2)}^{\{p_1,...,p_n\}}\{x_1, ..., x_n\} = \left(\bigoplus_{t_1}^{\{p_2,...,p_k\}}\{x_1, ..., x_k\}\right) \oplus_{p_1} \left(\bigoplus_{t_2}^{\{p_{k+1},...,p_n\}}\{x_k, ..., x_n\}\right).$$

For instance, given the binary tree $\mathsf{node}(\mathsf{node}(\mathsf{leaf}, \mathsf{leaf}), \mathsf{leaf}) \in T$, the list of elements $[x_0, x_1, x_2]$, and the coefficients $[p_1, p_2]$, we produce the tree $(x_1 \oplus_{p_1} x_2) \oplus_{p2} x_3$.

1.3 Symmetric tricocycloids

▶ **Definition 5** (Street, Garner). *A symmetric tricocycloid, $(H, \bullet, \circ, *)$, consists of a set H endowed with two binary functions, $(\bullet)\colon H \times H \to H$ and $(\circ)\colon H \times H \to H$, that are jointly invertible — meaning that the function $\varphi(p, q) = (p \bullet q, p \circ q)$ is a bijection — and a unary function, $(*)\colon H \to H$, satisfying the following axioms.*

1. $p \bullet (q \bullet r) = (p \bullet q) \bullet r$;
2. $(p \circ (q \bullet r)) \bullet (q \circ r) = (p \bullet q) \circ r$;
3. $(p \circ (q \bullet r)) \circ (q \circ r) = p \circ q$;
4. $p^{**} = p$;
5. $(p \bullet q)^* \bullet (p \circ q) = p \bullet q^*$;
6. $(p \bullet q)^* \bullet (p \circ q) = (p \circ q^*)^*$;

▶ **Proposition 6** (Kaddar [2], Lemma D.2). *The joint inverses of the two binary operations are given by $(\overset{*}{\bullet})\colon H \times H \to H$ and $(\overset{*}{\circ})\colon H \times H \to H$, defined by $(p \overset{*}{\bullet} q) = (p^* \bullet q^*)^*$ and $(p \overset{*}{\circ} q) = (p^* \circ q^*)^*$.*

▶ **Proposition 7** (Opposite tricocycloid). *Any symmetric tricocycloid induces an* opposite tricocycloid, $\mathrm{op}(H, \bullet, \circ, *) = (H, \overset{*}{\bullet}, \overset{*}{\circ}, *)$, *defined by $(p \overset{*}{\bullet} q) = (p^* \bullet q^*)^*$ and $(p \overset{*}{\circ} q) = (p^* \circ q^*)^*$.*

1.3.1 Symmetric tricocycloid morphisms

▶ **Definition 8.** *A* tricocycloid morphism, $f\colon (H_1, \bullet_1, \circ_1) \to (H_2, \bullet_2, \circ_2)$, *is a function satisfying* $f(p \bullet_1 q) = f(p) \bullet_2 f(q)$ *and* $f(p \circ_1 q) = f(p) \circ_2 f(q)$. *A* symmetric tricocycloid morphism, $f\colon (H_1, \bullet_1, \circ_1, *_1) \to (H_2, \bullet_2, \circ_2, *_2)$, *is a tricocycloid morphism moreover satisfying* $f(p^{*_1}) = f(p)^{*_2}$.

▶ Remark 9. The initial symmetric tricocycloid is given by the singleton; the terminal symmetric tricocycloid is given by the empty set.

1.3.2 Algebras of a tricocycloid

▶ **Definition 10** (Garner). *An algebra* $(X, \oplus)$ *for a symmetric tricocycloid,* $(H, \bullet, \circ, *)$, *consists of a set* X *and an operation* $(\oplus)\colon X \times H \times X \to X$ *satisfying the following axioms.*

1. $a \oplus_p a = a$;
2. $a \oplus_p b = b \oplus_{p^*} a$;
3. $(a \oplus_q b) \oplus_p c = a \oplus_{p \bullet q} (b \oplus_{p \circ q} c)$;

▶ **Theorem 11.** *Every symmetric tricocycloid,* $(H, \bullet, \circ, *)$, *is the carrier of an algebra over itself, defined as follows.*

$$(q_1 \oplus_p q_2)^* = (p \bullet q_1)^* \bullet (((p \circ q_1)^* \circ q_2^*) \bullet ((p \circ q_1)^* \bullet q_2^*)^*)^*$$

▶ **Example 12.** Convex spaces are algebras for the interval tricocycloid.

1.3.3 The rational interval is the free midpoint tricocycloid

▶ **Definition 13** (Midpoint tricycloid). *A* midpoint tricocycloid *is a tricocycloid endowed with an special element — the midpoint,* m *— that is a fixpoint for the antipode,* $m^* = m$. *A midpoint tricocycloid morphism is a tricocycloid morphism moreover preserving the midpoint,* $f(m) = m$.

▶ **Theorem 14.** *The rational interval is the free midpoint tricocycloid.*

1.3.4 Tricocycloids from multiplication and antipode

▶ **Theorem 15.** *Consider a group,* $(G, \cdot, e)$, *with a unary operation called the antipode,* $(-)^*\colon G \to G$, *satisfying the following two equations, where we write a fraction,* $\frac{x}{y}$, *to mean left division,* $y \backslash x$.

$$p^{**} = p; \qquad \left(\frac{pq^*}{(pq)^*}\right) = \frac{p^*}{(pq)^*}.$$

Then, any subset closed under antipodes, multiplication, and the comultiplication operation defined by $p \circ q = \frac{pq^*}{(pq)^*}$ *forms a symmetric tricocycloid.*

Proof. This proofs writes a fraction, $\frac{x}{y}$, to mean left division, $y \backslash x$. This makes the proof easier to read, but let us warn the reader that, while $a \cdot \frac{b}{a} = b$, in general $\frac{b}{a} \cdot a \neq b$.

Let us first show that multiplication and comultiplication are jointly invertible, with inverses given by $p \overset{*}{\bullet} q = (p^* \bullet q^*)^*$ and $p \overset{*}{\circ} q = (p^* \circ q^*)^*$.

$$(p \bullet q) \overset{*}{\bullet} (p \circ q) = \left((pq)^* \left(\frac{pq^*}{(pq)^*}\right)^*\right)^* = \left((pq)^* \frac{p^*}{(pq)^*}\right)^* = p^{**} = p;$$

$$(p \bullet q) \overset{*}{\circ} (p \circ q) = \left((pq)^* \circ \left(\frac{pq^*}{(pq)^*}\right)^*\right) = \left(\frac{(pq)^* \frac{pq^*}{(pq)^*}}{\left((pq)^* \frac{p^*}{(pq)^*}\right)^*}\right)^* = \left(\frac{pq^*}{p}\right)^* = q^{**} = q;$$

Taking antipodes, we also obtain $(p \overset{*}{\bullet} q) \bullet (p \overset{*}{\circ} q) = p$ and $(p \overset{*}{\bullet} q) \circ (p \overset{*}{\circ} q) = q$.

Let us now check the rest of the axioms of symmetric tricocycloids; of these, associativity of multiplication and involutivity of antipodes are given.

$$(p \circ (q \bullet r)) \bullet (q \circ r) = \frac{p(qr)^*}{(pqr)^*} \cdot \frac{qr^*}{(qr)^*} = \frac{pqr^*}{(pqr)^*} = (p \bullet q) \circ r;$$

$$(p \circ (q \bullet r)) \circ (q \circ r) = \frac{\frac{p(qr)^*}{(pqr)^*} \cdot \left(\frac{qr^*}{(qr)^*}\right)^*}{\left(\frac{p(qr)^*}{(pqr)^*} \cdot \frac{qr^*}{(qr)^*}\right)^*} = \frac{\frac{pq^*}{(pq)^*}}{\frac{(pq)^*}{(pqr)^*}} = \frac{pq^*}{(pq)^*} = p \circ q;$$

$$(p \bullet q)^* \bullet (p \circ q) = (pq)^* \frac{pq^*}{(pq)^*} = p \bullet q^*;$$

$$(p \bullet q)^* \circ (p \circ q) = \frac{(pq)^* \left(\frac{pq^*}{(pq)^*}\right)^*}{\left((pq)^* \cdot \frac{pq^*}{(pq)^*}\right)} = \frac{p^*}{(pq^*)^*} = \left(\frac{p^* q^{**}}{(pq^*)^*}\right)^* = (p \circ q^*)^*;$$

This concludes the proof. ◀

▶ **Corollary 16.** *The interval, the dual numbers, and the powerset are all tricocycloids.*

2 Commutative tricoycloids

▶ **Definition 17** (Commutative tricocycloid). *A* commutative tricocycloid *is a* symmetric tricocycloid*, $(H, \bullet, \circ, *)$, satisfying the following three additional equations.*

1. $p \bullet q = q \bullet p$;
2. $(p \circ q)^* \bullet q^* = (q \circ p)^* \bullet p^*$;
3. $(p \circ q)^* \circ q^* = ((q \circ p)^* \circ p^*)^*$.

▶ **Proposition 18.** *The interval tricocycloid is commutative.*

2.0.1 Not every tricocycloid is commutative

▶ **Proposition 19.** *Upper triangular matrices — of dimension $n \times n$ for some n — whose diagonal lies on the open interval form a symmetric tricocycloid, $U(n)$. In the 2-dimensional case, $U(2)$, the symmetric tricocycloid is defined by the following operations.*

$$\begin{pmatrix} x & a \\ 0 & y \end{pmatrix}^* = \begin{pmatrix} 1-x & a \\ 0 & 1-y \end{pmatrix};$$

$$\begin{pmatrix} x_1 & a_1 \\ 0 & y_1 \end{pmatrix} \bullet \begin{pmatrix} x_2 & a_2 \\ 0 & y_2 \end{pmatrix} = \begin{pmatrix} x_1 x_2 & x_1 a_2 + a_1 y_2 \\ 0 & y_1 y_2 \end{pmatrix};$$

$$\begin{pmatrix} x_1 & a_1 \\ 0 & y_1 \end{pmatrix} \circ \begin{pmatrix} x_2 & a_2 \\ 0 & y_2 \end{pmatrix} = \begin{pmatrix} \frac{x_1(1-x_2)}{1-x_1 x_2} & \varphi \\ 0 & \frac{y_1(1-y_2)}{1-y_1 y_2} \end{pmatrix};$$

Where we define φ as follows

$$\varphi = x_1(1-x_2)\frac{a_1 y_2 + a_2 x_1}{(1-x_1 x_2)(1-y_1 y_2)} + \frac{a_1(1-y_2) - a_2 x_1}{1-y_1 y_2}.$$

▶ **Corollary 20.** *Not every symmetric tricocycloid is commutative. The upper triangular tricocycloid $U(2)$ is not commutative.*

Proof. We simply provide an example of the non-commutativity of matrix multiplication satisfying all of the conditions of the upper triangular tricocycloid.

$$\begin{pmatrix} 1/2 & 1 \\ 0 & 1/3 \end{pmatrix} \bullet \begin{pmatrix} 1/3 & 1 \\ 0 & 1/2 \end{pmatrix} = \begin{pmatrix} 1/6 & 1 \\ 0 & 1/6 \end{pmatrix} \neq \begin{pmatrix} 1/6 & 2/3 \\ 0 & 1/6 \end{pmatrix} = \begin{pmatrix} 1/3 & 1 \\ 0 & 1/2 \end{pmatrix} \bullet \begin{pmatrix} 1/2 & 1 \\ 0 & 1/3 \end{pmatrix}$$

◀

References

1 Richard Garner. Abstract hypernormalisation, and normalisation-by-trace-evaluation for generative systems. *arXiv preprint arXiv:1811.02710*, 2018. arXiv:1811.02710.

2 Younesse Kaddar. Tricocycloids, effect monoids and effectuses. M2 Research Internship Report, Macquarie University, Department of Mathematics and Statistics, Sydney, Australia, 2019. Supervised by Richard Garner (April–August 2019). URL: https://younesse.net/assets/M2_report.pdf.

3 Ross Street. Fusion operators and cocycloids in monoidal categories. *Applied Categorical Structures*, 6(2):177–191, 1998.